Physics 250 Homework 2

Solutions

1. (*Problem 1 is written as a tutorial. For maximum benefit, write directly on this page.*) Imagine that you have measured a particular galaxy's ellipticity on each of two images, image A and image B. On image A you have measured e_A with uncertainty σ_A and on image B you have measured e_B with uncertainty σ_B. You wish to combine the two measurements into one estimate $\hat{e}$ of the true ellipticity e in an optimal way.

Clearly, $\hat{e}$ will involve some function of e_A and e_B. Let's just assume that e_A and e_B will enter only linearly. Write down $\hat{e}$ as a general linear combination of e_A and e_B without yet worrying about the values of the coefficients:

$$\hat{e} = ae_A + be_B$$

The estimation error $\tilde{e}$ is defined as the difference between $\hat{e}$ and e. We want $\hat{e}$ to be *unbiased*, that is, we want the expectation value $E[\tilde{e}]$ to be zero. Using the above definitions, write out the full expression for $E[\tilde{e}]$ and set it to zero:

$$E[\tilde{e}] = E[ae_A + be_B - e] = 0$$

Now, it is helpful to view e_A as e plus some random variable with zero mean. Likewise for e_B. Substitute that into your expression:

$$E[\tilde{e}] = E[a(e + \delta_A) + b(e + \delta_B) - e] = 0$$

Now, you should be able to evaluate the expectation value of each part of your expression, and simplify the result into a constraint on the coefficients of your linear combination:

$$E[\delta_A] = E[\delta_B] = 0$$

$$E[ae] = ae; E[be] = be; E[e] = e$$

$$\Rightarrow ae + be - e = 0$$

$$\Rightarrow a + b = 1$$

You have derived one constraint from the condition that your estimator be unbiased. We will now derive another from the condition that it be *optimal*. Write the expectation value of the squared error, $E[\tilde{e}^2]$:

$$E[\tilde{e}^2] = E[(\hat{e} - e)^2] = E[(ae_A + (1 - a)e_B - e)^2]$$

$$= E[(a(e + \delta_A) + (1 - a)(e + \delta_B) - e)^2]$$
$$= E[(a\delta_A + (1 - a)\delta_B)^2]$$
$$= E[a^2\delta_A^2 + (1 - a)^2\delta_B^2 + 2a(1 - a)\delta_A\delta_B]$$

To simplify, assume that images A and B are *uncorrelated*, that is, if δ_A and δ_B are the random parts of your measurement, $E[\delta_A\delta_B] = 0$.

$$E[\tilde{e}^2] = a^2\sigma_A^2 + (1 - a)^2\sigma_B^2$$

Take the derivative of this expression with respect to your unknown coefficient and set it to zero. This will minimize the mean square error and will result in an expression for the coefficients in your linear combination.

$$\frac{\partial E[\tilde{e}^2]}{\partial a} = 2a\sigma_A^2 - 2(1 - a)\sigma_B^2 = 0$$
$$\Rightarrow a\sigma_A^2 + a\sigma_B^2 = \sigma_B^2$$
$$\Rightarrow a = \frac{\sigma_B^2}{\sigma_A^2 + \sigma_B^2}$$
$$\Rightarrow b = \frac{\sigma_A^2}{\sigma_A^2 + \sigma_B^2}$$

Now that you know your coefficients, plug them back into your expression for $E[\tilde{x}^2]$ and derive a simple expression for $E[\tilde{x}^2]$.

$$E[\tilde{x}^2] = \frac{1}{\sigma_A^{-2} + \sigma_B^{-2}} \qquad \text{or} \qquad \frac{\sigma_A^2\sigma_B^2}{\sigma_A^2 + \sigma_B^2}$$

Finally, write down the full expression for $\hat{e}$:

$$\hat{e} = \frac{\sigma_B^2}{\sigma_A^2 + \sigma_B^2}e_A + \frac{\sigma_A^2}{\sigma_A^2 + \sigma_B^2}e_B$$

Comment on this expression. Does it make sense in the limit of equal measurement errors? In the limit of one measurement error being much smaller than the other?

In the case of equal measurement errors, the weights are equal, which makes sense. If one measurement error is much smaller than the other, $\hat{e}$ becomes nearly identical to the better measurement, which also makes sense.

Imagine that instead of your optimal estimator $\hat{e}$ you simply took an unweighted mean of your two measurements. Show that this estimator is unbiased. Again, it may help to write out e_A and e_B as true values plus random variables:

$$E[(e_A + e_B)/2 - e] = E[(e + \delta_A + e + \delta_B)/2 - e] = 0$$

What is the variance of this estimator? As before, assume that images A and B are uncorrelated.

$$E[((e_A + e_B)/2 - e)^2] = E[((e + \delta_A + e + \delta_B)/2 - e)^2]$$

$$= E[\delta_A^2/4 + \delta_B^2/4 + \delta_A\delta_B/2]$$
$$E[((e_A + e_B)/2 - e)^2] = E[\delta_A^2/4 + \delta_B^2/4]$$
$$= (\sigma_A^2 + \sigma_B^2)/4$$

What is the ratio of this variance to the variance in the optimal estimator?

$$\frac{1}{4}(\sigma_A^2 + \sigma_B^2)(\sigma_A^{-2} + \sigma_B^{-2}) \qquad \text{or} \qquad \frac{(\sigma_A^2 + \sigma_B^2)^2}{4\sigma_A^2\sigma_B^2}$$

Evaluate this ratio for $\sigma_A = \sigma_B$ and for $\sigma_A = 3\sigma_B$. Is the simpler estimator just slightly suboptimal?

$\sigma_A = \sigma_B$: ratio is 1, as expected

$\sigma_A = 3\sigma_B$: ratio is 100/36, so the simpler estimator is very suboptimal.

2. Now drop the assumption that the errors are uncorrelated. Let $E[\delta_A\delta_B] = p\sigma_A\sigma_B$, where p is a correlation coefficient which ranges from -1 to 1. Derive the optimal estimator and its variance. What happens when $p = \pm 1$? Why?

Some algebra leads to $a = \frac{\sigma_B^2 - p\sigma_A\sigma_B}{\sigma_A^2 + \sigma_B^2 - 2p\sigma_A\sigma_B}$ (where again $b = 1-a$) and $E[\tilde{e}^2] = \frac{\sigma_A^2\sigma_B^2(1-\rho^2)}{\sigma_A^2+\sigma_B^2-2p\sigma_A\sigma_B}$. When $p = \pm 1$, the variance vanishes. This reflects that fact that if you have two measurements with perfectly correlated errors, you can combine them in a way that cancels the error exactly. As an example, if you knew that a bit of noise in image A which increased a galaxy's e_1 estimate led to a corresponding bit of noise in image B which decreased *the same galaxy's e_1 estimate by the same amount (an admittedly unphysical situation), you could simply average the two estimates to cancel out the measurement errors perfectly.*

3. What if instead you drop the demand that the estimator be unbiased? Can you find an estimator which gives a lower variance? Is it practical?

In this case, you cannot eliminate the dependence on the true value e, so it does not lead to a practical estimator. If you knew something about the true value, you could *get lower variance by setting $\hat{e} = e$, which would give zero variance.*

4. Consider a 1-dimensional galaxy image with an exponential profile, $I = I_0 \exp(-x/x_0)$. You measure the intensities I_1 and I_2 at two points x_1 and x_2. The uncertainties σ_1 and σ_2 are not identical. Find an unbiased estimator for I_0 which is linear in I_1 and I_2. As in Problem 1, first find a form which is constrained to be unbiased, without worrying about what the coefficients are. Now make it optimal and derive the value of the coefficients. What is the mean square estimation error?

The condition that $\hat{I}_0$ be unbiased leads to

$$\hat{I}_0 = ae^{x_1}I_1 + (1-a)e^{x_2}I_2.$$

Minmizing the variance leads to

$$a = \frac{\sigma_2^2}{\sigma_1^2 e^{-2(x_2-x_1)} + \sigma_2^2}$$

and

$$E[(\hat{I_0} - I_0)^2] = \frac{1}{\frac{1}{\sigma_1^2} e^{-2x_1} + \frac{1}{\sigma_2^2} e^{-2x_2}}$$